Weak Zariski decompositions & the existence of minimal models, II

joint w/ N. Takamatsu

/$\mathbb{C}$, all varieties are normal & projective

Theorem A (L–Takamatsu 2021)

Assume the existence of minimal models for smooth varieties of dim $n-1$.

Let $(X/Z, \check{B}+\check{M})$ be a $\mathbb{Q}$-factorial NQC log canonical generalized pair of dimension n. Assume either:

(a) $(X, B+M)$ admits an NQC weak Zariski decomposition over Z, [in particular, K_X+B+M is pseudoeff.]

<u>OR</u>

(b) K_X+B+M is not pseudoeff. over Z.

Then for any $\mathbb{R}$-divisor P s.t. P is the pushforward of an NQC divisor over Z on some birational model of X, and for any $N \geq 0$ s.t. the g-pair $(X, (B+N)+(M+P))$ is log canonical and the divisor $K_X+B+N+M+P$ is nef over Z, then there exists a (K_X+B+M)-MMP with scaling of $P+N$ which terminates.

In particular, the generalized pair $(X, B+M)$ has either a minimal model $/Z$ or a Mori fibre space $/Z$.

Remark: This builds on and improves several previous fundamental results of Birkar, Han–Li, Moraga–Hacon, ...

Remark: In the statement of Theorem A there is an assumption

of $\mathbb{Q}$-factoriality. The reason:
we use recently developed foundational
results on the MMP of g-pairs
(the cone theorem, the contraction
theorem, the existence of flips)
of ~~Haco~~n-Liu $\rightsquigarrow$ they show these
results assuming $\mathbb{Q}$-factoriality.

<u>Theorem B</u>: (L - Tsakanikas - Xiaowei Jiang)

Let $(X/Z, B+M)$ be a $\mathbb{Q}$ factorial NQC log canonical g pair. Assume that $(X, B+M)$ has a <u>minimal model in the sense of Birkar - Shokurov over Z</u> OR that K_X+B+M is not pseudoeffective. Let A be an effective $\mathbb{R}$-Cartier divisor on X which is ample $/Z$ s.t. $K_X+B+M+A$ is nef $/Z$.

Then $\exists$ a (K_X+B+M)-MMP over Z with scaling of A which terminates.

In particular:

(a) $(X, B+M)$ has a min. model in the sense of Birkar–Shokurov iff it has a minimal model

$\not\!\!\!\text{ft}$.

(b) If K_X+B+M is not pseudoeffective, then $\exists$ a Mori fibre space of $(X, B+M)$.

Remark: • If (X, B) is a klt pair s.t. K_X+B is not pseudoeffective, then (X, B) has a Mfs by BCHM.

• If (X, B) is moreover log canonical, then the same is true by the work Hashimura–Hu.

• If $(X, B+M)$ is a g-pair s.t. $(X, 0)$ is klt, then it has a Mfs by Birkar – Zhang.

I want to discuss one of the main ideas of the proof of Thm B.

The proof relies on the following result:

Theorem C (Birkar, Han–Li, L–Tasin[?]) Let $(X, B+M)$ be a $\mathbb{Q}$-factorial NQC generalised pair, let P and N be as in Theorem A. (In particular $(X, (B+N)+(P+M))$ is log canonical and

$K_X+B+N+M+P$ is nef/Z.

Consider an MMP for K_X+B+M; with scaling of $P+N$ (over Z), denote by λ_i the nef thresholds in the steps of this MMP.

[we know: $\underline{\lambda_1 \geq \lambda_2 \geq \dots}$]

Set $\lambda := \lim_{j\to\infty} \lambda_j$.

If $\lambda \neq \lambda_j$ for every j and if $(X, (B+\lambda N)+(M+\lambda P))$ has a minimal model in the sense of Birkar–Shokurov, then this MMP terminates.

Sketch of the proof of Thm B

I will sketch the proof for one particular choice of $\underline{\underline{A}}$.
This uses an idea of Hashizume-Hu.

- The key is to choose carefully an ample divisor with which one wants to scale
- We may assume that this $(X, B+M)$-MMP consists only of flips
- Set $d := \dim_{\mathbb{R}} N^1(X/Z)_{\mathbb{R}}$
- for simplicity assume that B and M are $\mathbb{Q}$-divisors

- fix positive real numbers $\alpha_1, \dots, \alpha_d$ which are $\mathbb{Q}$-linearly independent
- pick general ample $\mathbb{Q}$-divisors $A^{(1)}, \dots, A^{(d)}$ over Z whose classes generate $N^1(X/Z)_{\mathbb{R}}$
- set $\boxed{A := \alpha_1 A^{(1)} + \dots + \alpha_d A^{(d)}}$
- run a (K_X+B+M)-MMP with scaling of A

$$
\begin{tikzcd}
(X_1, B_1+M_1) \arrow[rr, dashed] \arrow[dr, "\theta_1"'] & & (X_2, B_2+M_2) \arrow[rr, dashed] \arrow[dl, "\theta_1^+"] \arrow[dr, "\theta_2"'] & & (X_3, B_3+M_3) \arrow[dl, "\theta_2^+"] \\
& Z_1 & & Z_2 &
\end{tikzcd}
$$

Assume that this MMP does not terminate.

- denote by B_i, M_i, A_i and $A_i^{(k)}$ the pushforwards of B, M, A and $A^{(k)}$ on X_i
- for each i set

$$\lambda_i := \inf\{t \in \mathbb{R}_{\geq 0} \mid K_{X_i} + B_i + M_i + tA_i \text{ is nef}/Z\}$$

$$\Rightarrow \lambda_i \geq \lambda_{i+1} \quad \forall i$$

- By Theorem C it suffices to show that $\lambda_i > \lambda_{i+1}$ $\forall i$.

- Assume $\lambda_i = \lambda_{i+1}$ for some i

$$\begin{tikzcd}
 & (X_{i+1}, B_{i+1}+M_{i+1}) \arrow[dl, "\theta_i^+"'] \arrow[dr, "\theta_{i+1}"] & \\
Z_i & & Z_{i+1}
\end{tikzcd}$$

$\rightsquigarrow$ • pick a curve C on X_{i+1} which is contracted by θ_i^+

• pick a curve C' on X_{i+1} which is contracted by θ_{i+1}

• $\Rightarrow$ $(K_{X_{i+1}}+B_{i+1}+M_{i+1})\cdot C > 0$

&

$(K_{X_{i+1}}+B_{i+1}+M_{i+1})\cdot C' < 0$

$$\Rightarrow \beta := \frac{(K_{X_{i+1}} + B_{i+1} + M_{i+1}) \cdot C}{(K_{X_{i+1}} + B_{i+1} + M_{i+1}) \cdot C'}$$

$$\in \mathbb{Q} \cap (-\infty, 0)$$

$$\Rightarrow A_{i+1}^{(k)} \cdot (C - \beta C') \in \mathbb{Q}$$

$\forall\, k \in \{1, \dots, d\}$

- by construction of the $(K_X + B + M)$ MMP with scaling of A:

$$\underline{\underline{(K_{X_{i+1}} + B_{i+1} + M_{i+1} + \lambda_i A_{i+1})}} \cdot C = 0$$

and

$$(K_{X_{i+1}} + B_{i+1} + M_{i+1} + \lambda_{i+1} A_{i+1}) \cdot C' = 0$$

combining this with $\lambda_i = \lambda_{i+1}$:

$$\boxed{A_{i+1} \cdot (C - \beta C') = 0}$$

$\Downarrow$ replacing $A_{i+1} = \cdots$

$$\alpha_1 \underbrace{\boxed{A_{i+1}^{(1)} \cdot (C - \beta C')}}_{\in \mathbb{Q}} + \cdots + \alpha_d \underbrace{\boxed{A_{i+1}^{(d)} \cdot (C - \beta C')}}_{\in \mathbb{Q}} = 0$$

- since $\alpha_1, \dots, \alpha_d$ are $\mathbb{Q}$-lin. indep.:

$$\Rightarrow \underline{\underline{A_{i+1}^{(k)}}} \cdot (C - \beta C') = 0 \quad \forall k \in \{1, \dots, d\}$$

- since $A_{i+1}^{(k)}$ span $N'(X_{i+1}/Z)_{\mathbb{R}}$,

$$\Rightarrow C - \beta C' = 0 \text{ in } N_1(X_{i+1}/Z)_{\mathbb{R}}$$

- but $C - \beta C'$ is an effective curve
  (since $\beta < 0$)

A contradiction. $\square$

A few comments on the termination of flips:

- termination of flips in general is known in dim ≤ 3; (Shokurov, '85, Shokurov '96)
- 4-folds:
  - terminal fourfolds: KMM
  - canonical fourfolds: Fujino
  - for pairs (X,Δ) with $K_X+\Delta$ pseudoeffective: Moraga; Chen-Tsakanikas

starting point = existence of min. models

Logic: Assume that are MMP terminates. Then all MMPs terminate.

- These proofs rely on finding good invariants which drops after applying a flips ("difficulty")

- follow a different idea + flipping are unavoidable

- Birkar, Chen-Hacon, one uses special termination (Fujino, L-Moraga-Hacon) to show that if $K_X + \Delta =$

$= P + N$ → "negative part"

"positive part"

then any MMP terminates.

<u>Caveat:</u> The assumption in lower dimensions:

termination of all flips

⇓

in particular also for

$\mathbb{Q}$-pairs $(X, B+M')$ where

$(K_X + B + M)$ is <u>not pseudoeffective</u>

<u>Problem:</u> there is no strategy to attack the termination of flips for non-psef pairs!